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WEIGHTED L^p ESTIMATES FOR THE $\bar{\partial}$ -EQUATION ON CONVEX DOMAINS OF FINITE TYPE

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Abstract

We prove non-isotropic L^p ($1 \leq p \leq \infty$) estimates with weights for solutions of the Cauchy-Riemann equation on bounded convex domains of finite type in \mathbb{C}^n using the integral kernel method. We also give an example which guarantees the optimum of the estimates.

1. Introduction and statement of results

Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain of finite type m with a defining function ρ . In this paper we treat a certain weighted L^p estimates for the $\bar{\partial}$ -equation on Ω . The notation $\delta(\zeta)$ will stand for the distance from ζ to the boundary of Ω , $b\Omega$, which is up to constants $|\rho(\zeta)|$. With solutions using the integral kernel introduced by Cumenge [Cum01a] we can prove the following theorem.

Theorem 1.1. *For the domain Ω as above the equation $\bar{\partial}u = f$ has a solution u in Ω such that for $1 \leq p < \infty$*

$$(1) \quad \int_{\Omega} \delta(\zeta)^{\alpha-1} |u(\zeta)|^p dV(\zeta) \leq C_{p,\alpha} \int_{\Omega} \delta(\zeta)^{\alpha-1+p} \|f(\zeta)\|^p dV(\zeta), \quad \alpha > 0$$

and

$$(2) \quad \sup_{\zeta \in \Omega} \delta(\zeta)^{\alpha-1} |u(\zeta)| \leq C_{\alpha} \sup_{\zeta \in \Omega} \delta(\zeta)^{\alpha} \|f(\zeta)\|, \quad \alpha > 1,$$

if f is a smooth $(n, 1)$ -form with $\bar{\partial}f = 0$ and the right hand sides of (1) and (2) are finite. Here the non-isotropic norm of forms, $\|\cdot\| = \|\cdot\|_{\Omega}$ is defined by $\|f(\zeta)\| = \sup_{v \in \mathbb{C}^n \setminus \{0\}} |f(\zeta)(v)|/k(\zeta, v)$, where $k(\zeta, v)^{-1}$ is a weighted boundary distance of $\zeta \in \Omega$ in the direction v .

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Remark. (i) The non-isotropic norm $\|\cdot\|$ was first introduced by Bruna, Charpentier and Dupain [BCD98]. The definition of the quantity $k(\zeta, v)$ is rather complicate even though $k(\zeta, v)^{-1}$ is a natural weighted boundary distance, so we postpone the precise definition in the next section. (ii) Theorem 1.1 implies that for given a smooth $(0, 1)$ -form f with $\bar{\partial}f = 0$ one can find a solution for the $\bar{\partial}$ -equation on Ω and that solution satisfies the inequalities (1) and (2).

If Ω is strongly pseudoconvex, (1) and (2) were proved by Ahn-Cho [AC02] and Dautov-Henkin [DH79]. When the domain is convex of finite type, Cumenge [Cum01a] proved (1) in case $p = 1$. Therefore it seems natural to extend Cumenge's result to other L^p -norms, $1 \leq p \leq \infty$. There are a number of papers related to the $\bar{\partial}$ -equation on convex domains of finite type m . We mention a few of them, which are closely related to our work; Diederich-Fischer-Fornæss [DFF99], Cumenge [Cum01a] independently proved $1/m$ -Hölder estimates and Cumenge [Cum01a], Fischer [Fis01] obtained the best possible L^p estimates with respect to the isotropic norm. Diederich-Mazzilli [DM01] (resp. Cumenge [Cum01b]) obtained the characterization of the zero sets of functions in the Nevanlinna (resp. Nevanlinna-Djrbachian) classes using non-isotropic $L^1(b\Omega)$ (resp. weighted nonisotropic $L^1(\Omega)$) estimates for the solution of the $\bar{\partial}$ -equation on Ω .

Here we briefly sketch the methods used to prove Theorem 1.1. First, to obtain the solution of $\bar{\partial}$ on Ω we use an integral representation introduced by Berndtsson-Andersson [BA82]. In the integral representation the key part is a kernel with a weight containing the Bergman kernel of Ω . Second, for the estimates of the kernel and integrals relevant to the kernel we use non-isotropic polydiscs and ε -extremal coordinates of McNeal on Ω [McN94]. Last, in order to pass into L^p estimates from L^1 estimates we employ a variation of Hölder's inequality used in [AC02].

2. Preliminaries

2.1. Integral kernels and solution operators. Let $B(z, \zeta)$ be the Bergman kernel for the domain Ω . Then $B(z, \zeta)$ is holomorphic in z and antiholomorphic in ζ . Moreover $B(z, \zeta) \in C^\infty(\bar{\Omega} \times \bar{\Omega} \setminus \{(\zeta, \zeta), \zeta \in b\Omega\})$ and the boundary behavior of $B(z, \zeta)$ is now well understood by [McN94]. We define

$$Q = Q(z, \zeta) = \frac{1}{B(\zeta, \zeta)} \sum_{j=1}^n \left(\int_0^1 \frac{\partial B}{\partial z_j}(z_t, \zeta) dt \right) dz_j, \quad z_t = \zeta + t(z - \zeta).$$

For sufficiently large integer N to be determined and fixed later we define the weighed kernel on $(z, \zeta) \in \bar{\Omega} \times \Omega \setminus \{(\zeta, \zeta), \zeta \in b\Omega\}$,

$$\begin{aligned} K(z, \zeta) &= \sum_{k=0}^{n-1} c_{n,k,N} \left(\frac{B(z, \zeta)}{B(\zeta, \zeta)} \right)^{N-k} \frac{\partial_z |z - \zeta|^2 \wedge (\bar{\partial}_\zeta Q)^k \wedge (d\partial_z |\zeta - z|)^{n-k-1}}{|\zeta - z|^{2n-2k}} \\ &= \sum_{k=0}^{n-1} c_{n,k,N} K^{(k)}(z, \zeta), \end{aligned}$$

where $c_{n,k,N} = -(-1)^{n(n-1)/2} \binom{N}{n}$ and $K^{(k)}(z, \zeta)$ is the k -th term in the summation. For $f \in C^1_{(n,1)}(\Omega)$ with $\bar{\partial}f = 0$, if we define

$$(3) \quad u(z) = C \int_{\Omega} K(z, \zeta) \wedge f(\zeta), \quad z \in \Omega$$

then it is known that $\bar{\partial}u = f$ [Cum01a].

2.2. McNeal's result on the geometry of convex domains of finite type. We adapt to the notation of [Cum01a] and [McN94].

2.2.1. Weighted boundary distance. Define the radius of the largest complex disc centered at z in the direction v that fits in the domain $\{z : \rho(z) < \rho(\zeta) + \varepsilon\}$,

$$\tau(z, v, \varepsilon) = \sup\{r > 0 : |\rho(z + \lambda v) - \rho(z)| \leq \varepsilon, |\lambda| \leq r, \lambda \in \mathbb{C}\}.$$

Introduce a weighted boundary distance $k(z, v, \varepsilon) = \delta(z)/\tau(z, v, \varepsilon)$ and write $k(z, v)$ when $\varepsilon = \delta(z)/2$.

2.2.2. Non-isotropic polydisc. If $\{v_1, \dots, v_n\}$ is an ε -extremal basis of McNeal at z (see [McN94] and [BCD98] for the precise definition), then the non-isotropic polydisc at z with radius ε is defined by

$$P(z, \varepsilon) = \left\{ w = z + \sum_{j=1}^n w_j v_j, |w_j| \leq c\tau(z, v_j, \varepsilon) \right\},$$

where c is chosen so that $w \in P(z, \varepsilon)$ implies $|\rho(w) - \rho(z)| < \varepsilon$. The properties of τ which were proved by McNeal [McN94] are the following:

$$(4) \quad \tau(z, v_1, \varepsilon) \approx \varepsilon \lesssim \tau(z, v_n, \varepsilon) \leq \dots \leq \tau(z, v_2, \varepsilon) \lesssim \varepsilon^{\frac{1}{m}}$$

and for $0 < \varepsilon_1 \leq \varepsilon_2$

$$(5) \quad \left(\frac{\varepsilon_1}{\varepsilon_2} \right) \tau(z, v, \varepsilon_2) \lesssim \tau(z, v, \varepsilon_1) \lesssim \left(\frac{\varepsilon_1}{\varepsilon_2} \right)^{\frac{1}{m}} \tau(z, v, \varepsilon_2).$$

Proposition 2.1 ([McN94]).

- (i) For all $C > 0$, $\text{vol } P(z, C\varepsilon) \approx \text{vol } P(z, \varepsilon)$ uniformly in z, ε with constants depending on C .
- (ii) $\text{vol } P(z, \varepsilon) \approx \text{vol } P(\zeta, \varepsilon)$ if $P(z, \varepsilon) \cap P(\zeta, \varepsilon) \neq \emptyset$.
- (iii) $\tau(\zeta, v, \varepsilon) \approx \tau(z, v, \varepsilon)$ for $\zeta \in P(z, \varepsilon)$.
- (iv) If $\{v_1, \dots, v_n\}$ is an ε -extremal basis at z , then we have $\text{vol } P(z, \varepsilon) \approx \prod_{i=1}^n \tau(z, v_i, \varepsilon)^2$.

2.2.3. Tent and quasi distance. For $z \in \overline{\Omega}$ close to the boundary and $\eta > 0$, $T(z, \eta) = P(\pi(z), \eta) \cap \Omega$ is called the tent at z of radius η , where $\pi(z)$ is the projection of z to the boundary. The quasi distance of McNeal is defined as follows:

$$\mathcal{M}(z, \zeta) \approx \mathcal{M}(\zeta, z) = \inf\{\eta : \zeta \in P(z, \eta)\}$$

for $|\zeta - z| \ll 1$ and z close to $b\Omega$. For $v \in \mathbb{C}^n$ with $|v| = 1$, $\varphi \in C^\infty(\Omega)$ let $D_v \varphi$ denote the directional derivatives of φ in the direction v . From now on set $\eta = \eta(z, \zeta) = |\rho(z)| + |\rho(\zeta)| + \mathcal{M}(z, \zeta)$. Then the following proposition is proved by McNeal [McN94].

Proposition 2.2. For every $p \in b\Omega$ there exists a neighborhood U of p such that for $\zeta, z \in U \cap \Omega$, $\mu, \nu \in \mathbb{N}$, $v, v' \in \mathbb{C}^n$ with $|v| = |v'| = 1$,

- (i) $|D_v^\mu \overline{D}_{v'}^\nu B(z, \zeta)| \leq C(\mu, \nu) \tau(\zeta, v, \eta)^{-\mu} \tau(\zeta, v', \eta)^{-\nu} (\text{vol } T_{\zeta, z})^{-1}$. Here $\text{vol } T_{\zeta, z}$ is the volume of the smallest tent containing both z, ζ .
- (ii) For $\zeta \in U \cap \Omega$, $B(\zeta, \zeta) \geq (\text{vol } P(\zeta, \delta))^{-1}$, $\delta = \delta(\zeta) = |\rho(\zeta)|/2$.

2.2.4. Coverings. There exists a constant $\beta > 1$ such that

$$\mathcal{M}(z, \zeta) < \varepsilon \Rightarrow \zeta \in P(z, \beta\varepsilon), \quad z \in U \cap \Omega, \quad 0 < \varepsilon \ll 1,$$

where U is some neighborhood defined in Proposition 2.2. For the integral estimates we define the covering of $W \cap \Omega$

$$\mathcal{C}_0(z) = P(z, \beta d(z)) \cap W \cap \Omega,$$

$$\mathcal{C}_\ell(z) = \{\zeta \in \Omega \cap W : 2^{\ell-1} d(z) \leq \mathcal{M}(z, \zeta) < 2^\ell d(z)\}, \quad \ell \geq 1,$$

where $W = 1/2U$.

3. Kernel estimates

3.1. Estimate of the term $K^{(k)}(z, \zeta)$, $0 \leq k \leq n-1$. Now we want to write all forms with respect to extremal coordinates of McNeal at ζ or z . Let $\{e_j^{(\ell)} = e_j^{(\ell)}(\zeta), 1 \leq j \leq n\}$ be a $\beta 2^\ell \delta$ -extremal basis at ζ and $(e_j^{(\ell)}(z))_j$ a $\beta 2^\ell d$ -extremal basis at z , respectively. If $v_j = v_j^{(\ell)}(\zeta)$ is the

j -th component of their coordinates, we denote $L_j^{(\ell)} = \partial/\partial v_j$ and $L_j^{(\ell)*}$ which is the dual of $L_j^{(\ell)}$. To simplify notations, in any ambiguous case, we write $L_j^{(z)}$, $\bar{L}_j^{(\zeta)}$, $L_j^{*(z)}$, $\bar{L}_j^{*(\zeta)}$ for $L_j^{(\ell)(z)}$, $\bar{L}_j^{(\ell)(\zeta)}$, $L_j^{(\ell)*}(z)$, $\bar{L}_j^{(\ell)*}(\zeta)$, $1 \leq j \leq n$, where the superscripts z , ζ mean the derivations act on the variables z , ζ , respectively. To save us from confusion, we denote $\text{dist}(\zeta, \Omega)$ and $\text{dist}(z, b\Omega)$ by $\delta = \delta(\zeta)$ and $d = d(z)$, respectively.

First we estimate $K^{(k)}(z, \zeta) \wedge f(\zeta)$, $1 \leq k \leq n-1$. Let $\mathcal{R} = \int_0^1 \partial_z B(z_t, \zeta) dt$. Then computing the k -th exterior product we obtain

$$(\bar{\partial}_\zeta Q)^k = c_k \frac{\bar{\partial}_\zeta B(\zeta, \zeta)}{B(\zeta, \zeta)^{k+1}} \wedge \mathcal{R} \wedge (\bar{\partial}_\zeta \mathcal{R})^{k-1} + \frac{(\bar{\partial}_\zeta \mathcal{R})^k}{B(\zeta, \zeta)^k}.$$

Using this and expressing all forms in terms of $\bar{L}_j^{(\zeta)}$ and $L_j^{(z)}$'s, we have

$$K^{(k)}(z, \zeta) \wedge f(\zeta) = \frac{B(z, \zeta)^{N-k}}{B(\zeta, \zeta)^N} \frac{1}{|\zeta - z|^{2n-2k}} [H_1 + H_2],$$

where

$$\begin{aligned} H_1 &= \frac{1}{B(\zeta, \zeta)} \sum_{I, J} L_{i_0}^{(z)} |\zeta - z|^2 (\bar{L}_{j_1}^{(\zeta)} B)(\zeta, \zeta) \int_0^1 (L_{i_1}^{(z)} B)(z_t, \zeta) dt \\ &\quad \times \prod_{\nu=2}^k \left(\int_0^1 (\bar{L}_{j_\nu}^{(\zeta)} L_{i_\nu}^{(z)} B)(z_t, \zeta) dt \right) [f(\zeta)(e_{j_n}^{(\ell)})] \\ &\quad \times A_{IJS}(z, \zeta) L_S^{*(\zeta)} \wedge \bar{L}_J^{*(\zeta)} \wedge L_I^{*(z)} \\ H_2 &= \sum_{I, J} L_{i_0}^{(z)} |\zeta - z|^2 \prod_{\nu=1}^k \left(\int_0^1 (\bar{L}_{j_\nu}^{(\zeta)} L_{i_\nu}^{(z)} B)(z_t, \zeta) dt \right) [f(\zeta)(e_{j_n}^{(\ell)})] \\ &\quad \times A_{IJS}(z, \zeta) L_S^{*(\zeta)} \wedge \bar{L}_J^{*(\zeta)} \wedge L_I^{*(z)}. \end{aligned}$$

Here $I = I(k) = I_k \cup I_k'$, $I_k = \{i_0, i_1, \dots, i_k\}$ and $I_k' = \{i_{k+1}, \dots, i_{n-1}\}$; $J = J(k) = J_k \cup J_k'$, $J_k = \{j_1, j_2, \dots, j_k\}$ and $J_k' = \{j_{k+1}, \dots, j_n\}$, $i_\nu, j_\nu \in S = \{1, 2, \dots, n\}$, $L_I^{*(z)} = L_{i_0}^{*(z)} \wedge \dots \wedge L_{i_{n-1}}^{*(z)}$, etc. and A_{IJS} is uniformly bounded on $\Omega \times \Omega$. Note that if $k = 1$, then the product term $\prod_{\nu=2}^k (\dots)$ of H_1 does not appear.

Next we note the following inequality

$$(6) \quad |f(\zeta)(e_j^{(\ell)}(\zeta))| \leq \|f(\zeta)\| \frac{\delta(\zeta)}{\tau(\zeta, e_j^{(\ell)}(\zeta), \delta)} \lesssim \|f(\zeta)\|.$$

Then it is easy to see that

$$\begin{aligned} |K^{(0)}(z, \zeta) \wedge f(\zeta)| &\leq \|f(\zeta)\| |K^{(0)}(z, \zeta)| \\ &\leq \|f(\zeta)\| \left(\frac{|B(z, \zeta)|}{B(\zeta, \zeta)} \right)^N \frac{1}{|\zeta - z|^{2n-1}}. \end{aligned}$$

First we estimate $|B(z, \zeta)|/B(\zeta, \zeta)$, $\int_0^1 L_{i_1}^{(z)} B(z_t, \zeta) dt$ and $\int_0^1 \overline{L}_{j_n}^{(\zeta)} L_{i_\nu}^{(z)} B(z_t, \zeta) dt$ on the neighborhood U of $p \in b\Omega$, where U is the neighborhood defined in Proposition 2.2. By Proposition 2.2 (ii), (4) and (5) the following estimates can be proved:

$$\begin{aligned} \frac{|B(z, \zeta)|}{B(\zeta, \zeta)} &\leq \frac{\delta(\zeta)}{\eta(z, \zeta)} \\ \left| \int_0^1 L_{i_1}^{(z)} B(z_t, \zeta) dt \right| &\lesssim [\text{vol } P(\zeta, \delta) \tau(\zeta, e_{i_1}^{(\ell)}, \delta)]^{-1} \\ \left| \int_0^1 \overline{L}_{j_n}^{(\zeta)} L_{i_\nu}^{(z)} B(z_t, \zeta) dt \right| &\lesssim [\text{vol } P(\zeta, \delta) \tau(\zeta, e_{j_n}^{(\ell)}, \delta) \tau(\zeta, e_{i_\nu}^{(\ell)}, \delta)]^{-1}. \end{aligned}$$

(For details see [Cum01a].) Combining above estimates and the inequality (6) we have for $1 \leq k \leq n-1$

$$\begin{aligned} &|K^{(k)}(z, \zeta) \wedge f(\zeta)| \\ &\lesssim \sum_{I_k, J_k} \frac{\delta(\zeta)^{N-k}}{\eta(z, \zeta)^{N-k}} \frac{\delta(\zeta) \|f(\zeta)\|}{|\zeta - z|^{2n-2k-1}} \frac{1}{\prod_{\nu=1}^k \tau(\zeta, e_{j_\nu}^{(\ell)}, \delta) \tau(\zeta, e_{i_\nu}^{(\ell)}, \delta)} \frac{1}{\tau(\zeta, e_{j_n}^{(\ell)}, \delta)} \end{aligned}$$

and

$$|K^{(0)}(z, \zeta) \wedge f(\zeta)| \lesssim \frac{\delta(\zeta)^{N-1}}{\eta(z, \zeta)^N} \frac{\delta(\zeta) \|f(\zeta)\|}{|\zeta - z|^{2n-1}}.$$

For the simplification of notation we write for $z, \zeta \in U$

$$\begin{aligned} K_+^{(0)}(z, \zeta) &= \frac{\delta(\zeta)^{N-1}}{\eta(z, \zeta)^N} \frac{1}{|\zeta - z|^{2n-1}} \\ K_+^{(k)}(z, \zeta) &= \frac{\delta(\zeta)^{N-k}}{\eta(z, \zeta)^{N-k}} \frac{1}{|\zeta - z|^{2n-2k-1}} \\ &\quad \times \frac{1}{\prod_{\nu=1}^k \tau(\zeta, e_{j_\nu}^{(\ell)}, \delta) \tau(\zeta, e_{i_\nu}^{(\ell)}, \delta)} \frac{1}{\tau(\zeta, e_{j_n}^{(\ell)}, \delta)}. \end{aligned}$$

3.2. Estimates of $K_+^{(k)}(z, \zeta)$ on coverings $\{\mathcal{C}_\ell(z)\}$ and $\{\mathcal{C}_\ell(\zeta)\}$.

We only consider the estimates of $K_+^{(k)}(z, \zeta)$ for $k = 1, n-1$. In the integral estimates other cases can be reduced to cases $k = 1, n-1$. If $k = 0$, we can directly estimate integrals on the neighborhood W . First assume that $\zeta \in U$ is fixed and we will estimate $K_+^{(k)}(z, \zeta)$ on $\mathcal{C}_\ell(\zeta)$. By the definition of $\mathcal{C}_\ell(\zeta)$, $\mathcal{M}(z, \zeta)$ and (5), it is easy to see that

$$\begin{aligned} \eta(z, \zeta) &\approx 2^\ell \delta(\zeta), & z \in \mathcal{C}_\ell(\zeta) \\ \tau(\zeta, e_j^{(\ell)}(\zeta), \delta) &\gtrsim 2^{-\ell} \tau(\zeta, e_j^{(\ell)}(\zeta), \beta 2^\ell \delta). \end{aligned}$$

Note that $\tau_{j_n}^{(\ell)}(\zeta) \leq \tau_2^{(\ell)}(\zeta)$. Therefore we have for $z \in \mathcal{C}_\ell(\zeta)$

$$(7) \quad K_+^{(n-1)}(z, \zeta) \lesssim \frac{\tau_2^{(\ell)}(\zeta)}{2^{(N-3n+2)\ell} |\zeta - z| \prod_{j=1}^n \tau_j^{(\ell)}(\zeta)^2}$$

$$(8) \quad K_+^{(1)}(z, \zeta) \lesssim \sum_{I_1', J_1'} \frac{\tau_{i_0}^{(\ell)} \prod_{\nu=2}^{n-1} \tau_{i_\nu}^{(\ell)}(\zeta) \tau_{j_\nu}^{(\ell)}(\zeta)}{2^{(N-4)\ell} \prod_{j=1}^n \tau_j^{(\ell)}(\zeta)^2 |\zeta - z|^{2n-3}},$$

where $\tau_j^{(\ell)}(\zeta) = \tau(\zeta, e_j^{(\ell)}(\zeta), \beta 2^\ell \delta(\zeta))$. Next assume that $z \in U$ is fixed and we will estimate $K^{(k)}(z, \zeta)$ on $\mathcal{C}_\ell(z)$. By Proposition 2.1 (ii) and (iii), we have

$$\tau(\zeta, e_j^{(\ell)}, \delta) \gtrsim \left(\frac{\delta}{2^\ell d} \right) \tau(\zeta, e_j^{(\ell)}, \beta 2^\ell \delta) \approx \left(\frac{\delta}{2^\ell d} \right) \tau(z, e_j^{(\ell)}(z), \beta 2^\ell d).$$

Since $\eta(z, \zeta) \approx 2^\ell d(z)$, $\zeta \in \mathcal{C}_\ell(z)$ we have

$$(9) \quad K_+^{(n-1)}(z, \zeta) \lesssim \left(\frac{\delta}{2^\ell d} \right)^{N-3n+2} \frac{1}{|\zeta - z|} \frac{\tau_2^{(\ell)}(z)}{\prod_{j=1}^n \tau_j^{(\ell)}(z)^2}$$

and

$$\begin{aligned} (10) \quad K_+^{(1)}(z, \zeta) &\lesssim \sum_{\substack{i, j, p=1 \\ i \neq p}}^n \left(\frac{\delta}{2^\ell d} \right)^{N-4} \frac{1}{|\zeta - z|^{2n-3}} \frac{1}{\tau_j^{(\ell)}(z) \tau_i^{(\ell)}(z) \tau_p^{(\ell)}(z)} \\ &\lesssim \sum_{I_1', J_1'} \left(\frac{\delta}{2^\ell d} \right)^{N-4} \frac{1}{|\zeta - z|^{2n-3}} \frac{\tau_{i_0}^{(\ell)}(z) \prod_{\nu=2}^{n-1} \tau_{i_\nu}^{(\ell)}(z) \tau_{j_\nu}^{(\ell)}(z)}{\prod_{j=1}^n \tau_j^{(\ell)}(z)^2}. \end{aligned}$$

4. Integral estimates

In this section we verify preliminary integral estimates that are an essential step to prove our Theorem 1.1.

Lemma 4.1. *Let $\alpha > 0$ and $\varepsilon > 0$ with $\alpha - 1 - \varepsilon > -1$. Then for $k = 0, 1, \dots, n-1$ we have*

$$(11) \quad \int_{\Omega} d(z)^{\alpha-1-\varepsilon} K_+^{(k)}(z, \zeta) dV(z) \leq C_{\alpha, \varepsilon} \delta(\zeta)^{\alpha-\varepsilon-1}$$

$$(12) \quad \int_{\Omega} \delta(\zeta)^{-\varepsilon} K_+^{(k)}(z, \zeta) dV(\zeta) \leq C_{\varepsilon} d(z)^{-\varepsilon}.$$

Proof: We prove (11) and (12) for $k = 0, 1, n-1$. The other cases can be reduced to the cases $k = 0, 1, n-1$. Since the only singularity is of the form $|\zeta - z|^{-j}$, we may assume that $z, \zeta \in W$. W can be covered by $\cup_{\ell} \mathcal{C}_{\ell}(z)$ and $\cup_{\ell} \mathcal{C}_{\ell}(\zeta)$ so basically we have to deal with the domain of the form $\mathcal{C}_{\ell}(z)$ or $\mathcal{C}_{\ell}(\zeta)$.

(i) By the estimate (7) we have for any integer $\ell \geq 0$

$$(13) \quad \int_{\mathcal{C}_{\ell}(\zeta)} d(z)^{\alpha-1-\varepsilon} K_+^{(n-1)}(z, \zeta) dV(z) \\ \lesssim \frac{\tau_2^{(\ell)}(\zeta)}{2^{(N-3n+2)\ell} \prod_{j=1}^n \tau_j^{(\ell)}(\zeta)^2} \int_{P(\zeta, \beta 2^{\ell} \delta)} \frac{d(z)^{\alpha-1-\varepsilon}}{|\zeta - z|} dV(z).$$

To obtain a desired estimate, we use the system of coordinate associated to the basis $(e_1^{(\ell)}(\zeta), \dots, e_n^{(\ell)}(\zeta))$. We set

$$(14) \quad w_k = \langle \zeta - z, e_k^{(\ell)} \rangle, \quad 1 \leq k \leq n, \quad t_1 = -\rho(z), \quad t_2 = \operatorname{Im} w_1$$

and for $2 \leq k \leq n$,

$$t_{2k-1} = \operatorname{Re} w_k, \\ t_{2k} = \operatorname{Im} w_k.$$

Since $\tau_1^{(\ell)}(\zeta) \approx 2^\ell \delta$ and $d(z) \lesssim 2^\ell d$ for $z \in P(\zeta, \beta 2^\ell d)$, by the coordinates change (14) we have

$$\begin{aligned}
 (15) \quad & \int_{P(\zeta, \beta 2^\ell \delta)} \frac{d(z)^{\alpha-1-\varepsilon}}{|\zeta - z|} dV(z) \\
 & \lesssim \int_{\substack{|t_j| < 2^\ell \delta, j=1,2 \\ |w_j| < \tau_j^{(\ell)}, j \geq 2}} \frac{t_1^{\alpha-1-\varepsilon} dt_1 dt_2 dV(w_2, \dots, w_n)}{|w_2|} \\
 & \lesssim \frac{1}{\alpha - \varepsilon} (2^\ell \delta)^{\alpha-\varepsilon+1} \int_{|w_j| < \tau_j^{(\ell)}, j \geq 2} \frac{dV(w_2, \dots, w_n)}{|w_2|} \\
 & \lesssim \frac{1}{\alpha - \varepsilon} (2^\ell \delta)^{\alpha-\varepsilon-1} \prod_{j=1}^n \frac{\tau_j^{(\ell)}(\zeta)^2}{\tau_2^{(\ell)}(\zeta)}.
 \end{aligned}$$

If we choose an integer N so that $N - 3n + 3 - \alpha > 0$, then (13) and (15) give

$$\begin{aligned}
 & \int_{W \cap \Omega} d(z)^{\alpha-1-\varepsilon} K_+^{(n-1)}(z, \zeta) dV(z) \\
 & \lesssim \sum_{\ell} 2^{-(N-3n+3-\alpha+\varepsilon)\ell} \delta(\zeta)^{\alpha-\varepsilon-1} \lesssim \delta(\zeta)^{\alpha-\varepsilon-1}.
 \end{aligned}$$

To prove (11) for $k = 1$ we have to consider the integral

$$\int_{P(\zeta, \beta 2^\ell \delta)} \frac{d(z)^{\alpha-1-\varepsilon}}{|\zeta - z|^{2n-3}} dV(z).$$

To change coordinates we again use the coordinates (14). Here we may assume that $i_0 < i_1$, $\nu = \min(i_0, j_1)$, $\mu = \max(i_0, j_1)$ and $r_{i_1} = |w_{i_1}|$. We first consider $t_1, t_{2\nu}, t_{2i_2-1}, t_{2i_2}$ variables and then we integrate with the remaining $(2n-4)$ variables, t' . Since $\tau_1^{(\ell)} \approx 2^\ell \delta$ and $2^\ell \delta \lesssim \tau_\mu^{(\ell)}$, we

have

$$\begin{aligned}
(16) \quad & \int_{P(\zeta, \beta 2^\ell \delta)} \frac{d(z)^{\alpha-1-\varepsilon}}{|\zeta - z|^{2n-3}} dV(z) \\
& \lesssim \int \cdots \int_{\substack{|t_{2k-1}| + |t_{2k}| < \tau_k^{(\ell)} \\ k=1, \dots, n}} \frac{t_1^{\alpha-1-\varepsilon} dt_1 \cdots dt_{2n}}{|t|^{2n-3}} \\
& \lesssim \int_{\substack{|t_1| < \tau_1^{(\ell)} \\ |t_{2\nu}| < \tau_\nu^{(\ell)}}} t_1^{\alpha-1-\varepsilon} dt_1 dt_{2\nu} \int_{\substack{r_{i_1} < \tau_{i_1}^{(\ell)} \\ |t'| < 1}} \frac{r_{i_1} dr_{i_1} dV(t')}{(r_{i_1} + |t'|)^{2n-3}} \\
& \lesssim \frac{1}{\alpha - \varepsilon} (2^\ell \delta)^{\alpha-1-\varepsilon} \tau_\nu^{(\ell)} \tau_\mu^{(\ell)} \int_{|t'| < 1} \int_0^{\tau_{i_1}^{(\ell)}} \frac{r_{i_1} dr_{i_1} dV(t')}{(r_{i_1} + |t'|)^{2n-3}} \\
& \lesssim \frac{1}{\alpha - \varepsilon} (2^\ell)^{\alpha-1-\varepsilon} (\delta)^{\alpha-1-\varepsilon} \tau_{i_0}^{(\ell)} \tau_{i_1}^{(\ell)} \tau_{j_1}^{(\ell)}.
\end{aligned}$$

From the estimates (8) and (16), since $N - 3 - \alpha > 0$ we see that

$$\begin{aligned}
& \int_{W \cap \Omega} d(z)^{\alpha-1-\varepsilon} K_+^{(1)}(z, \zeta) dV(z) \\
& \lesssim \sum_{\ell} 2^{-(N-3-\alpha+\varepsilon)\ell} \delta(\zeta)^{\alpha-1-\varepsilon} \lesssim \delta(\zeta)^{\alpha-1-\varepsilon}.
\end{aligned}$$

To prove (11) for $k = 0$ we note that

$$\begin{aligned}
\eta(z, \zeta) & \approx |\rho(\zeta)| + |\rho(z)| + \mathcal{M}(z, \zeta) \\
& \gtrsim |\rho(\zeta)| + |\rho(z)| + |\zeta_1 - z_1|, \quad z, \zeta \in U \cap \Omega.
\end{aligned}$$

Here the coordinates (z_1, \dots, z_n) are ε -extremal coordinates of McNeal at ζ . Hence there is a system of coordinates $t = (t_1, \dots, t_{2n})$ on $U \cap \Omega$ such that $t_1 = -\rho(z)$, $t_2 = \text{Im}(\zeta_1 - z_1)$ and $t'(\zeta) = (t_3(\zeta), \dots, t_{2n}(\zeta)) = 0$. From the definition of $K_+^{(0)}(z, \zeta)$ we have

$$\begin{aligned}
(17) \quad & \int_{W \cap \Omega} d(z)^{\alpha-1-\varepsilon} K_+^0(z, \zeta) dV(z) \\
& \lesssim \delta(\zeta)^{N-1} \int_{|t| < 1} \frac{|t_1|^{\alpha-1-\varepsilon} dt_1 \cdots dt_{2n}}{(|t_1| + |t_2| + |\rho(\zeta)|)^N (|t_1 - \rho(\zeta)| + |t_2| + |t'|)^{2n-1}}.
\end{aligned}$$

Introducing polar coordinates with respect to the variables t' we have

$$\begin{aligned} I_1(\zeta) &= \int_{|t|<1} \frac{|t_1|^{\alpha-1-\varepsilon} dt_1 \cdots dt_{2n}}{(|t_1| + |t_2| + |\rho(\zeta)|)^N (|t_1 - \rho(\zeta)| + |t_2| + |t'|)^{2n-1}} \\ &\lesssim \int_{|(t_1, t_2)|<1} \frac{|t_1|^{\alpha-1-\varepsilon} dt_1 dt_2}{(|t_1| + |t_2| + |\rho(\zeta)|)^N (|t_1 - \rho(\zeta)| + |t_2|)}. \end{aligned}$$

If we make the change of variables $t_1 = |\rho(\zeta)|t'_1$ and $t_2 = |\rho(\zeta)|t'_2$ and omit the primes, then we obtain

$$(18) \quad I_1(\zeta) \lesssim |\rho(\zeta)|^{-N+\alpha-\varepsilon} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|t_1|^{\alpha-1-\varepsilon} dt_1 dt_2}{(|t_1| + |t_2| + 1)^N (|t_1 - 1| + |t_2|)}.$$

If $\alpha - 1 - \varepsilon \geq 0$ then

$$\begin{aligned} J_1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|t_1|^{\alpha-1-\varepsilon} dt_1 dt_2}{(|t_1| + |t_2| + 1)^N (|t_1 - 1| + |t_2|)} \\ &\lesssim \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dt_1 dt_2}{(|t_1| + |t_2| + 1)^{N-\alpha+1+\varepsilon} (|t_1 - 1| + |t_2|)} \\ &\lesssim \int_0^{\infty} \int_0^{\infty} \frac{dt_1 dt_2}{(|t_1| + |t_2| + 2)^{N-\alpha+1+\varepsilon} (|t_1| + |t_2|)} \\ &\lesssim \int_0^{\infty} \frac{s ds}{s(s+2)^{N-\alpha+1+\varepsilon}} \lesssim 1 \quad \text{if } N - \alpha > 0. \end{aligned}$$

If $-1 < \alpha - 1 - \varepsilon < 0$, then we have for $0 < \gamma < 1$

$$\begin{aligned} J_1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dt_1 dt_2}{|t_1|^{1-\alpha+\varepsilon} (|t_1| + |t_2| + 1)^N (|t_1 - 1| + |t_2|)} \\ &\lesssim \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dt_1 dt_2}{|t_1|^{1-\alpha+\varepsilon} (|t_1| + |t_2| + 1)^N |t_1 - 1|^\gamma |t_2|^{1-\gamma}} \\ &\lesssim \int_{-\infty}^{\infty} \frac{dt_1}{|t_1|^{1-\alpha+\varepsilon} |t_1 - 1|^\gamma (|t_1| + 1)^{\frac{N}{2}}} \int_{-\infty}^{\infty} \frac{dt_2}{|t_2|^{1-\gamma} (|t_2| + 1)^{\frac{N}{2}}}. \end{aligned}$$

Since $0 < \gamma < 1$ and $0 < 1 - \alpha + \varepsilon < 1$, it is easy to see that

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{dt_1}{|t_1|^{1-\alpha+\varepsilon} |t_1 - 1|^\gamma (|t_1| + 1)^{\frac{N}{2}}} \\ &\lesssim \int_0^{\infty} \frac{dt_1}{t_1^{1-\alpha+\varepsilon} (t_1 + 1)^{\frac{N}{2}}} + \int_0^{\infty} \frac{dt_1}{t_1^\gamma (t_1 + 1)^{\frac{N}{2}}} < \infty. \end{aligned}$$

Since J_1 is bounded, by the inequalities (17) and (18), we have

$$\int_{W \cap \Omega} d(z)^{\alpha-1-\varepsilon} K_+^0(z, \zeta) dV(z) \lesssim \delta(\zeta)^{\alpha-1-\varepsilon}.$$

(ii) By the estimate (9) we have

$$\begin{aligned} & \int_{\mathcal{C}_\ell(z)} \delta(\zeta)^{-\varepsilon} K_+^{(n-1)}(z, \zeta) dV(\zeta) \\ & \lesssim \frac{\tau_2^{(\ell)}(z)}{(2^\ell d)^{N-3n+2}} \frac{1}{\prod_{j=1}^n \tau_j^{(\ell)}(z)^2} \int_{P(z, \beta 2^\ell d)} \frac{\delta(\zeta)^{N-3n+2-\varepsilon}}{|\zeta - z|} dV(\zeta). \end{aligned}$$

We introduce new coordinates associated to the basis $(e_1^{(\ell)}(z), \dots, e_n^{(\ell)}(z))$ and set

$$(19) \quad u_k = \langle \zeta - z, e_k^{(\ell)} \rangle, \quad 1 \leq k \leq n, \quad t_1 = -\rho(\zeta), \quad t_2 = \operatorname{Im} u_2$$

and for $2 \leq k \leq n$,

$$\begin{aligned} t_{2k-1} &= \operatorname{Re} u_k, \\ t_{2k} &= \operatorname{Im} u_k. \end{aligned}$$

The same calculation as (i) shows that

$$\int_{P(z, \beta 2^\ell d)} \frac{\delta(\zeta)^{N-3n+2-\varepsilon}}{|\zeta - z|} dV(\zeta) \lesssim (2^\ell d)^{N-3n+2-\varepsilon} \frac{\prod_{j=1}^n \tau_j^{(\ell)}(z)^2}{\tau_2^{(\ell)}(z)}, \quad \varepsilon > 0.$$

Hence we have for $\varepsilon > 0$

$$\int_{W \cap \Omega} \delta(\zeta)^{-\varepsilon} K_+^{(n-1)}(z, \zeta) dV(\zeta) \lesssim \sum_{\ell} (2^\ell)^{-\varepsilon} d^{-\varepsilon} \lesssim d(z)^{-\varepsilon}.$$

By the estimates (10) we have

$$\begin{aligned} & \int_{\mathcal{C}_\ell(z)} \delta(\zeta)^{-\varepsilon} K_+^{(1)}(z, \zeta) dV(\zeta) \\ & \lesssim \sum_{I, J} \frac{\tau_{i_0}^{(\ell)}(z) \prod_{\nu=2}^{n-1} \tau_{i_\nu}^{(\ell)}(z) \tau_{j_\nu}^{(\ell)}(z)}{(2^\ell d)^{N-4} \prod_{j=1}^n \tau_j^{(\ell)}(z)^2} \int_{P(z, \beta 2^\ell d)} \frac{\delta(\zeta)^{N-4-\varepsilon}}{|\zeta - z|^{2n-3}} dV(\zeta). \end{aligned}$$

Using the coordinate system (19) and estimating the integral as (16) we obtain

$$\int_{P(z, \beta 2^\ell d)} \frac{\delta(\zeta)^{N-4-\varepsilon}}{|\zeta - z|^{2n-3}} dV(\zeta) \lesssim (2^\ell d)^{N-4-\varepsilon} \tau_{i_0}^{(\ell)}(z) \tau_{i_1}^{(\ell)}(z) \tau_{j_1}^{(\ell)}(z).$$

Thus as before we have for $\varepsilon > 0$

$$\int_{W \cap \Omega} \delta(\zeta)^{-\varepsilon} K_+^{(1)}(z, \zeta) dV(\zeta) \lesssim \sum_{\ell} (2^\ell)^{-\varepsilon} d^{-\varepsilon} \lesssim d(z)^{-\varepsilon}.$$

Finally if $k = 0$ then we have

$$\int_{U \cap \Omega} \delta(\zeta)^{-\varepsilon} K_+^{(0)}(z, \zeta) dV(\zeta) \lesssim \int_{U \cap \Omega} \frac{\delta(\zeta)^{N-1-\varepsilon}}{(\eta(z, \zeta))^N} \frac{1}{|\zeta - z|^{2n-1}} dV(\zeta).$$

Using a system of coordinates with respect to an ε -extremal basis at $z \in U$ we have

$$\eta(z, \zeta) \gtrsim |\rho(z)| + |\rho(\zeta)| + |\zeta_1 - z_1|.$$

Using the coordinates $t_1 = -\rho(\zeta)$, $t_2 = \text{Im}(\zeta_1 - z_1)$ and $t' = (t_3, \dots, t_{2n})$ satisfying $t'(z) = 0$ we have

$$\begin{aligned} I_2(z) &= \int_{U \cap \Omega} \frac{\delta(\zeta)^{N-\varepsilon-1}}{(\eta(z, \zeta))^N} \frac{1}{|\zeta - z|^{2n-1}} dV(\zeta) \\ &\lesssim \int_{|t| < 1} \frac{|t_1|^{N-\varepsilon-1} dt_1 dt_2 dV(t')}{(|t_1 - |\rho(z)|| + |t_2| + |t'|)^{2n-1} (|t_1| + |t_2| + |\rho(z)|)^N}. \end{aligned}$$

The right hand side of above inequality has the same type integration as J_1 . Thus the same method can be applied to obtain $I_2(z) \lesssim d(z)^{-\varepsilon}$. \square

5. Proof of Main Theorem

Now we come to the final step in the proof of Main Theorem. Let $u(z)$ be a solution of $\bar{\partial}u = f$ in (3). Then by the definition of $K_+^{(k)}(z, \zeta)$, $k = 0, 1, \dots, n-1$, we have for each $z \in \Omega$

$$(20) \quad |u(z)| \lesssim \sum_{k=0}^{n-1} \int_{\Omega} \delta(\zeta) \|f(\zeta)\| K_+^{(k)}(z, \zeta) dV(\zeta).$$

Therefore to complete the proof of Theorem 1.1, it suffices to show that

$$\begin{aligned} (21) \quad \int_{\Omega} d(z)^{\alpha-1} dV(z) \left[\int_{\Omega} \delta(\zeta) \|f(\zeta)\| K_+^{(k)}(z, \zeta) dV(\zeta) \right]^p \\ \lesssim \int_{\Omega} \delta(\zeta)^{\alpha} \|f(\zeta)\|^p dV(\zeta). \end{aligned}$$

Cumenge [Cum01a] have already proved the inequality (21) in case $p = 1$, so we do not repeat the proof. First assume that $1 < p < \infty$ and fix $\alpha > 0$ and p . We choose a sufficiently large integer N so that

$N - 3n + 3 - \alpha > 0$. Let $q > 1$ be a positive real such that $1/p + 1/q = 1$. Then by Hölder's inequality and Lemma 4.1 (12) we have

$$\begin{aligned} \int_{\Omega} \delta(\zeta) \|f(\zeta)\| K_+^{(k)}(z, \zeta) dV(\zeta) &= \int_{\Omega} \left(\delta \|f\| (K_+^{(k)})^{1/p} \delta^\varepsilon \right) \left((K_+^{(k)})^{1/q} \delta^{-\varepsilon} \right) dV \\ &\leq \left\{ \int_{\Omega} \delta^{p+\varepsilon p} \|f\|^p K_+^{(k)} \right\}^{\frac{1}{p}} \left\{ \int_{\Omega} K_+^{(k)} \delta^{-\varepsilon q} \right\}^{\frac{1}{q}} \\ &\lesssim d^{-\varepsilon} \left\{ \int_{\Omega} \delta^{p+\varepsilon p} \|f\|^p K_+^{(k)} \right\}^{\frac{1}{p}}. \end{aligned}$$

We apply Fubini's theorem to the left hand side of (21) to obtain

$$\begin{aligned} \int_{\Omega} d(z)^{\alpha-1} dV(z) \left[\int_{\Omega} \delta(\zeta) \|f(\zeta)\| K_+^{(k)}(z, \zeta) dV(\zeta) \right]^p \\ \leq \int_{\Omega} \|f(\zeta)\|^p \delta(\zeta)^{p+\varepsilon p} dV(\zeta) \left[\int_{\Omega} d(z)^{\alpha-1-\varepsilon p} K_+^{(k)}(z, \zeta) dV(z) \right] \\ \leq \int_{\Omega} \|f(\zeta)\|^p \delta(\zeta)^{p+\varepsilon p} \cdot \delta(\zeta)^{\alpha-1-\varepsilon p} dV(\zeta) \\ = \int_{\Omega} \delta(\zeta)^{\alpha-1+p} \|f(\zeta)\|^p dV(\zeta) \end{aligned}$$

by Lemma 4.1 (11). Here we choose $\varepsilon > 0$ so small that $\alpha - 1 - \varepsilon p > -1$. Next we prove the inequality for $p = \infty$. Again by the relation (20), it suffices to show that for all $z \in \Omega$,

$$d(z)^{\alpha-1} \int_{\Omega} \delta(\zeta) \|f(\zeta)\| K_+^{(k)}(z, \zeta) dV(\zeta) \leq \sup_{\zeta \in \Omega} \delta(\zeta)^\alpha \|f(\zeta)\|, \quad \alpha > 1.$$

By the Lemma 4.1 (12) we see that

$$\begin{aligned} d(z)^{\alpha-1} \int_{\Omega} \delta(\zeta) \|f(\zeta)\| K_+^{(k)}(z, \zeta) dV(\zeta) \\ \lesssim \sup_{\zeta \in \Omega} (\delta(\zeta)^\alpha \|f(\zeta)\|) \left[d(z)^{\alpha-1} \int_{\Omega} \delta(\zeta)^{-\alpha+1} K_+^{(k)}(z, \zeta) dV(\zeta) \right] \\ \lesssim \sup_{\zeta \in \Omega} (\delta(\zeta)^\alpha \|f(\zeta)\|) [d(z)^{\alpha-1} \cdot d(z)^{-\alpha+1}] \\ \lesssim \sup_{\zeta \in \Omega} \delta(\zeta)^\alpha \|f(\zeta)\|, \end{aligned} \quad \text{for } \alpha > 1.$$

6. Example

In this section we give an example to show that the estimates in Main Theorem are sharp in some sense at the cases $2 \leq p < \infty$. Let $E_m = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = |z_1|^2 + |z_2|^m - 1 < 0\}$, where m is an even number. Then E_m is a convex domain of finite type m . If $1 \leq p < \infty$ and $0 < \alpha < \infty$ we define a non-isotropic L^p space with weight α , $L_{\alpha}^p(E_m, \|\cdot\|)$ that consists of all $(0, 1)$ -form f satisfying

$$\|f\|_{p, \alpha, E_m}^p = \int_{E_m} \|f(z)\|^p |\rho(z)|^{\alpha-1+p} dV < \infty$$

and L^p space with weight α , $L_{\alpha}^p(\Omega)$ that consists of all measurable functions g satisfying

$$\|g\|_{p, \alpha}^p = \int_{E_m} |g(z)|^p |\rho(z)|^{\alpha-1} dV < \infty.$$

Now we can prove the following theorem.

Theorem 6.1. *For each $p \geq 2$ and α , there exists a $\bar{\partial}$ -closed $(0, 1)$ -form $f \in L_{\gamma}^p(E_m, \|\cdot\|)$, for all $\gamma > \alpha$, or $f \in L_{\alpha}^r(E_m, \|\cdot\|)$, for all $r < p$, such that no solution to $\bar{\partial}u = f$ belongs to $L_{\alpha}^p(E_m)$.*

Proof: Fix $p \geq 2$ and α . Put $f(z_1, z_2) = d\bar{z}_2/(1 - z_1)^d$, where $dp - \alpha - p/m - 2/m + 1 = 2$. Then f is a $\bar{\partial}$ -closed $(0, 1)$ -form on E_m . For simplicity of notation, we let $b_{z_1} = (1 - |z_1|^2)^{1/m}$. Then by the definition of $\|\cdot\|$, we have $\|f(z_1, z_2)\| \lesssim (1 - |z_1|^2 - |z_2|^m)^{1/m-1}/|1 - z_1|^d$. It follows that

$$(22) \quad \|f\|_{p, \gamma, E_m}^p \lesssim \int_{|z_1| < 1} \frac{dA(z_1)}{|1 - z_1|^{dp}} \int_{|z_2| < b_{z_1}} (1 - |z_1|^2 - |z_2|^m)^{\gamma-1+p/m} dA(z_2).$$

Using polar coordinate change, we have

$$(23) \quad \begin{aligned} I(z_1) &= \int_{|z_2| < b_{z_1}} (1 - |z_1|^2 - |z_2|^m)^{\gamma+p/m-1} dA(z_2) \\ &= 2\pi \int_0^{b_{z_1}} (1 - |z_1|^2 - r^m)^{\gamma+p/m-1} r dr \\ &= 2\pi (1 - |z_1|^2)^{\gamma+p/m-1+2/m} \int_0^1 (1 - s^m)^{\gamma+p/m-1} s ds \\ &\lesssim (1 - |z_1|^2)^{\gamma+p/m-1+2/m}, \end{aligned}$$

where we set $s = r/(1 - |z_1|^2)^{1/m}$. To calculate the upper bound of (22) we need the following lemma:

Lemma 6.2 ([Rud80]). *For $z \in B_n = \{z \in \mathbb{C}^n : |z| < 1\}$, c real, $\eta > -1$, define*

$$J_{c,\eta}(z) = \int_{B_n} \frac{(1 - |\zeta|^2)^\eta}{|1 - \bar{\zeta} \cdot z|^{n+1+\eta+c}} dV(\zeta).$$

When $c < 0$, then $J_{c,\eta}$ is bounded in B_n . When $c > 0$, then $J_{c,\eta}(z) \approx (1 - |z|^2)^{-c}$. Finally, $J_{0,\eta} \approx -\log(1 - |z|^2)$.

From (22), (23) and by Lemma 6.2 it follows that if $\gamma > \alpha$, then

$$\begin{aligned} \|f\|_{p,\gamma,E_m}^p &\lesssim \int_{|z_1|<1} \frac{dA(z_1)}{|1 - z_1|^{dp-\gamma+1-p/m-2/m}} \\ &= \lim_{r \rightarrow 1^-} \int_{|z_1|<1} \frac{dA(z_1)}{|1 - z_1 r|^{dp-\gamma+1-p/m-2/m}} \lesssim 1 \end{aligned}$$

since $dp - \gamma + 1 - p/m - 2/m < 2$. Let $v(z_1, z_2) = \bar{z}_2/(1 - z_1)^d$. Then it is clear that $\bar{\partial}v = f$ on E_m . On the other hand, we have

$$(24) \quad \|v\|_{p,\alpha}^p = \int_{|z_1|<1} \frac{dA(z_1)}{|1 - z_1|^{dp}} \int_{|z_2|<b_{z_1}} |z_2|^p (1 - |z_1|^2 - |z_2|^m)^{\alpha-1} dA(z_2).$$

By polar coordinate change, we have

$$\begin{aligned} J(z_1) &= \int_{|z_2|<b_{z_1}} |z_2|^p (1 - |z_1|^2 - |z_2|^m)^{\alpha-1} dA(z_2) \\ (25) \quad &= 2\pi(1 - |z_1|^2)^{\alpha-1} \int_0^{b_{z_1}} r^{p+1} \left(1 - \frac{r^m}{1 - |z_1|^2}\right)^{\alpha-1} dr \\ &= 2\pi(1 - |z_1|^2)^{\alpha-1+\frac{(p+2)}{m}} \int_0^1 (1 - s^m)^{\alpha-1} s^{p+1} ds \\ &\gtrsim (1 - |z_1|^2)^{\alpha-1+\frac{(p+2)}{m}}. \end{aligned}$$

From (24) and (25) we have

$$\begin{aligned}
 \|v\|_{p,\alpha}^p &\gtrsim \int_{|z_1|<1} \frac{(1-|z_1|^2)^{\alpha-1+\frac{(p+2)}{m}}}{|1-z_1|^{dp}} dA(z_1) \\
 (26) \quad &= \lim_{r \rightarrow 1^-} \int_{|z_1|<1} \frac{(1-|z_1|^2)^{\alpha-1+\frac{(p+2)}{m}}}{|1-z_1 r|^{dp}} dA(z_1) \\
 &\approx \lim_{r \rightarrow 1^-} \log \left(\frac{1}{1-r^2} \right) = \infty,
 \end{aligned}$$

by Lemma 6.2. Thus, $v \notin L_\alpha^p(B_2)$.

Next we consider the inner product $\langle h, v \rangle_\alpha$ for every $h \in L_\alpha^2(B_2) \cap \mathcal{O}(B_2)$. By Fubini's theorem, we have

$$\begin{aligned}
 \langle h, v \rangle_\alpha &= \int_{E_m} h(\zeta) \overline{v(\zeta)} |\rho(\zeta_1, \zeta_2)|^{\alpha-1} dV(\zeta) \\
 &= \int_{|\zeta_1|<1} \frac{dA(\zeta_1)}{(1-\bar{\zeta}_1)^d} \int_{|\zeta_2|<b_{\zeta_1}} \zeta_2 h(\zeta_1, \zeta_2) (1-|\zeta_1|^2-|\zeta_2|^m)^{\alpha-1} dA(\zeta_2).
 \end{aligned}$$

Putting $\zeta_2 = re^{i\theta}$, we see

$$\begin{aligned}
 &\int_{|\zeta_2|<b_{\zeta_1}} \zeta_2 h(\zeta_1, \zeta_2) (1-|\zeta_1|^2-|\zeta_2|^m)^{\alpha-1} dA(\zeta_2) \\
 &= \int_0^{b_{\zeta_1}} \int_0^{2\pi} r^2 e^{i\theta} h(\zeta_1, re^{i\theta}) (1-|\zeta_1|^2-r^m)^{\alpha-1} d\theta dr \\
 &= \int_0^{b_{\zeta_1}} r^2 (1-|\zeta_1|^2-r^m)^{\alpha-1} \left(\int_0^{2\pi} e^{i\theta} h(\zeta_1, re^{i\theta}) d\theta \right) dr \\
 &= \int_0^{b_{\zeta_1}} r^2 (1-|\zeta_1|^2-r^m)^{\alpha-1} \cdot 0 \cdot dr = 0,
 \end{aligned}$$

since $h(\zeta_1, \cdot)$ is holomorphic. Thus v is orthogonal to $L_\alpha^2(B_2) \cap \mathcal{O}(E_m)$, i.e., v is the canonical solution for $\bar{\partial}u = f$. To complete our theorem we need another well-known theorem on the boundedness of the weighted Bergman projections on E_m .

Proposition 6.3 ([Cho], [LS92]). *Let $\mathbb{B}_\alpha: L_\alpha^2(E_m) \rightarrow L_\alpha^2(E_m) \cap \mathcal{O}(E_m)$ be the orthogonal projection, $\alpha > 0$. Then $\mathbb{B}_\alpha: L_\alpha^p(E_m) \rightarrow L_\alpha^p(E_m) \cap \mathcal{O}(E_m)$ is a bounded operator for every $1 < p < \infty$.*

Assume that $\bar{\partial}u = f \in L_\gamma^p(E_m, \|\cdot\|)$ and $u \in L_\alpha^p(E_m)$, $\gamma > \alpha$. Then by the Proposition 6.3, $v = u - \mathbb{B}_\alpha(u)$ and it would be in $L_\alpha^p(E_m)$.

By (26) this is impossible. Hence there is no solution u in $L^p_\alpha(E_m)$ to the equation $\bar{\partial}u = f$.

If $r < p$, i.e. $dr - \alpha - r/2 < 2$, then it also follows by a similar calculation to the above that $f \in L^r_\alpha(E_m, \|\cdot\|)$ and no solution u to $\bar{\partial}u = f$ belongs to $L^p_\alpha(E_m)$. \square

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